Effect of Randoms on Signal-to-Noise Ratio in TOF PET

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Abstract—Time-of-flight (TOF) positron emission tomography (PET) has the potential to improve signal-to-noise ratio (SNR). In the literature, this improvement has been estimated as proportional to the square root of $D/\Delta x$, where $D$ is the radial dimension of the object to be imaged, and $\Delta x$ is the spatial uncertainty associated with the time resolution of the scanner. This estimate does not include the effect of the randoms. In this paper we discuss the role of random coincidence in TOF PET; in particular, the improvement in SNR due to random reduction in TOF PET. We introduce and discuss a simple model for this improvement, modify the traditional estimate for SNR gain, and compare it to experimental data from a Siemens Hi-Rez PET scanner with TOF capability.

Index Terms—PET, randoms, signal-to-noise ratio, time-of-flight, TOF.

I. INTRODUCTION

Twenty years after the first studies on time-of-flight (TOF) positron emission tomography (PET), new fast detectors that utilize lutetium orthosilicate (LSO) and lanthanum bromide ($\text{LaBr}_3$) have renewed the interest in this technique.

The introduction of LSO in PET, with its relative high light yield (about 30,000 photons/MeV), high Z and density (effective Z is 66 and density 7.4 g/cm$^3$), and short decay time (40 ns), has changed the landscape of PET [1]. Time resolutions down to 300 ps full-width at half-maximum (FWHM) have been measured in coincidence with small LSO crystals, making TOF PET attractive [2]. Preliminary results using a LSO scanner with coarse time resolution (1.2 ns) have already shown the feasibility of such an approach [3].

PET scanners based on LaBr$_3$ have also been recently proposed [4],[5]. Lanthanum bromide has very high light output (about 60,000 photons/MeV), fast decay time (35 ns), and acceptable Z and density (effective Z is 41 and density 5.3 g/cm$^3$). Time resolutions as low as 165 ps has been measured for a LaBr$_3$ (30%Ce) single crystal in coincidence with a BaF$_2$ crystal[6].

A first advantage of this improved time resolution is the feasibility of narrow time coincidence windows. The coincidence window can be easily reduced to 4–4.5 ns, corresponding to the TOF difference across the scanner field of view (FOV). This has the direct effect of reducing the random coincidences (random count rates are linear with the coincidence window width), and therefore the noise in the image. The improvement to the noise variance (or the NEC) obtained through a reduction in the time coincidence window has been shown in the literature [7]. Even if it were technically possible to further reduce the coincidence window (with faster detectors), true coincidence events coming from the area at the edge of the FOV would be rejected. It is possible to go beyond this limit and further increase random rejection only tailoring the coincidence window based on some a priori knowledge of the patient position in the FOV.

The second opportunity opened by the new fast detectors is the feasibility of TOF PET. TOF reconstruction algorithms promise large improvement in image SNR. The localization of the annihilation point along the line of TOF allows the reconstruction algorithm to filter out coincidence events that have an inconsistent TOF difference. This has a direct positive effect on the noise variance (the square root of the inverse of the signal-to-noise ratio (SNR)) of the resulting image. Variance reduction due to TOF reconstruction has been estimated from the 1980s, using simple and effective reasoning, as proportional to $\Delta x/D$, where $\Delta x$ is the time resolution of the system expressed in spatial units and $D$ is the size of the object to be imaged [8], [9].

In this paper we will elaborate on this reasoning, starting with the approach by Brownell and Strother, who relate the SNR in the conventional (or non-TOF) PET image to the square root of the noise equivalent count rate (NEC):

$$\text{SNR} = \sqrt{\text{NEC}} = \text{const} \cdot n^{-1/2} \cdot \left(\frac{T^2}{T + S + R}\right)^{1/2} \quad (1)$$

where $T$ represents the total true in the image, $S$ scatter events, $R$ random coincidences, and $n$ is the number of image elements contributing to a projection of the sinogram [10], [11]. In the case of a uniform distribution of activity in a cylinder of diameter $D$, and where $d$ is the size of the image element, $n = D/d$. The expression between the parentheses is commonly referred to as NEC.

In fact, the derivation of (1) assumes several constraints. We assume a uniform cylinder with uniform distribution of activity, located in the center of the FOV. Also, the acquisition system and the correction algorithms provide independent noiseless estimates of scatter and random coincidences.

If TOF is incorporated, the image elements $n$ contributing to the projection of a sinogram in (1) are limited to the image elements identified by the measured TOF difference and some neighboring elements, $n = \Delta x/d$, where $\Delta x$ is the localization uncertainty $c\Delta t/2$ related to the time resolution $\Delta t$. In other words, the image elements outside the localization window are filtered out and do not contribute to the projection.
If we assume that the NEC term for SNR in TOF images is the same as for conventional images, the SNR expressions for TOF and conventional images differ only for the value of \( \eta \). Using the appropriate values of \( \eta \) in (1), the SNR improvement due to TOF reconstruction can therefore be estimated in first approximation as:

\[
\text{SNR}_{\text{TOF}} \cong \sqrt{\frac{D}{\Delta x}} \cdot \text{SNR}_{\text{conv}}.
\]

Equation (2) [12] is commonly used to estimate the SNR gain due to TOF, which can be expressed as \( \sqrt{D/\Delta x} \). Such gain is higher for systems with better time resolution, and for bigger objects.

With all its accuracy limits and approximations, the gain expressed in (2) is still considered a good estimate of the theoretical TOF gain in SNR. Experimental evidence of a TOF gain in SNR dates back to the first generation of TOF PET [13],[14], and simulation papers have recently investigated such TOF gain [15],[16],[17],[18]. New evidence is reported today in an upcoming generation of TOF PET scanners [3],[5]. We expect that when these new TOF PET scanners are available, more extensive studies will properly assess the validity and limits of (2).

Still, it has already been observed that (2) does not account for the different TOF distribution of true, scatter, and random coincidences[7], since it is assumed that the variance contribution due to true, scatter, and random coincidences decreases by the same factor. This is not a realistic approximation.

In this paper, we propose a simple variation of (2), which includes a functional dependence of such gain from the random fraction. We also present some experimental data to support the revised expression for TOF-induced SNR gain.

II. SNR AND RANDOMS IN TOF RECONSTRUCTION

It has been argued in the literature that (2) does not properly account for random and scatter contributions to the variance reduction [7]. The intuitive reasoning is that, if the variance reduction for contribution from true coincidences is proportional to the ratio \( \Delta x/D \), where \( D \) is the size of the object to be imaged, the variance reduction for scatter and random coincidences should follow similar behavior. But the size \( D_S \) of the scatter distribution is larger than the physical object, and the size \( D_R \) of the random distribution is as large as the field-of-view (FOV) of the scanner. Thus, (2) is an underestimation of the real variance reduction due to TOF in the presence of scatter and randoms.

Experimental observation and attempts to model for the effect of randoms on the TOF gain date back to the first generation of TOF PET scanners, when different contributions by true and random counts to variance in the image were mentioned [19]. Recently, Monte Carlo simulation also showed that the SNR TOF gain increases with the random fraction [15].

In our approach, we will start from (1) and assume the same general constraints on the geometry (we use a cylindrical uniform distribution of activity, and we want to estimate the SNR at the center of the object). We will also assume the same constraints on the experimental setup and data acquisition (independent noiseless estimates of scatter and random coincidences).

But this time, we do not assume that the NEC term in (1) is the same for TOF and conventional reconstruction.

In fact, while the true events are confined within the original volume of diameter \( D \), the scatter distribution extends beyond the volume and is not uniform.

The randoms are uniformly distributed in the sinogram and, in order to keep the model simple, we assume that the randoms are uniformly distributed also in the image space, on the whole FOV of diameter \( D_{\text{FOV}} \). In this case \( D_{\text{FOV}} \) is not defined by the physical dimensions of the tunnel of the scanner, but by the time coincidence window \( D_{\text{FOV}} = c t_w/2 \). The reduction factor in variance \( \Delta x/D \) estimated for trues is therefore different for the other two components. In other words, the NEC part in (1) has a different form in conventional and TOF reconstruction, and it does not simply cancel in the ratio.

In the Appendix, we derive the two different forms for the variance in the conventional and TOF cases. In such derivations, we use the reasonable approximations that the trues are uniformly distributed within the cylinder of diameter \( D \), that the randoms are uniformly distributed within the cylinder of diameter \( D_{\text{FOV}} \), and that the scatter is also uniformly distributed within the cylinder of diameter \( D \). The last approximation is confirmed by the fact that the scatter does not extend outside the physical cylinder as much as the randoms. It is also confirmed by simulations that show very weak dependence of SNR gain on scatter fraction [15].

According to the Appendix, the SNR takes two different forms, expressed as:

\[
\text{SNR}^2(\text{conv}) = \text{const} \cdot \frac{T^2}{T + S + (D/D_{\text{FOV}}) \cdot R}
\]

\[
\text{and}
\]

\[
\text{SNR}^2(\text{TOF}) = \text{const} \cdot \frac{D}{\Delta x} \cdot \frac{T^2}{T + S + (D/D_{\text{FOV}})^2 \cdot R}.
\]

In these equations, \( D_{\text{FOV}} \) is the diameter of the scanner FOV as defined by the coincidence time window \( t_w \) (\( D_{\text{FOV}} = c t_w/2 \)), where \( \Delta x \) is the positional uncertainty due to time resolution and \( D \) is the diameter of the patient or object to be imaged. After defining the fraction \( \beta = D/D_{\text{FOV}} \) (patient size over FOV diameter), the SNR final gain can therefore be expressed as:

\[
\text{SNR}_{\text{TOF}} = \sqrt{\frac{D}{\Delta x}} \cdot \sqrt{\frac{T + S + \beta R}{T + S + \beta^2 R}} \cdot \text{SNR}_{\text{conv}}
\]

or, in terms of the random fraction \( R_f = R/(T+S) \):

\[
\text{SNR}_{\text{TOF}} = \sqrt{\frac{D}{\Delta x}} \cdot \sqrt{\frac{1 + \beta R_f}{1 + \beta^2 R_f}} \cdot \text{SNR}_{\text{conv}}
\]

This corresponds to a more precise estimate of the SNR gain using TOF and, for a given patient size, it is a function of the
random fraction. Equation (6) converges to (2) at low random fraction values. Equation (2) is therefore a low random fraction estimate of the TOF gain.

Since the data are distributed on several TOF bins, techniques such as random smoothing are recommended in order to obtain a “noiseless” estimate of the randoms. If this is not possible, notice that a corresponding formula for the SNR can be obtained replacing $R$ with $2R$ in equations (3), (4), (5), and (6).

In the following sections, some experimental data to support the model introduced in this paper are presented.

III. MATERIAL AND METHODS

A Siemens Hi-Rez scanner was used for this study [20]. The Hi-Rez is not a scanner designed for TOF, but it is able to deliver TOF information for each detected event. It has a time resolution of about 1.2 ns, which allows the observation of TOF improvement. The Hi-Rez is a three-ring scanner composed of 48 blocks per ring. Each block is made of $13 \times 13$ LSO crystals, $4 \times 4 \times 20$ mm$^3$, and is associated with four photomultiplier tubes (PMTs). Data are conventionally organized in sinograms with 336 radial bins (0.2-cm size), 336 angular bins, and 313 planes. The planes are organized into five segments, and 81 are direct planes. Data were acquired in list mode and rebinned offline.

The TOF information is coded in 4 bits plus a sign bit, and each time bin is 500 ps. The coincidence time window is 4.5 ns, corresponding to 67.5 cm, which is also the physical FOV of the scanner. Nine TOF bins are therefore sufficient to cover the whole FOV.

The system was time aligned using the method described in reference [3], and the time resolution of the system was measured with a 20-cm line source in the radial center of the FOV. A list mode file was acquired, and prompt and random coincidence events were separated. Histograms of the TOF information were generated for randoms and net trues (prompts minus randoms). We estimated the time resolution of the system (1.28 ns) as the FWHM of the Gaussian that fit the net trues TOF histogram.

In Fig. 1, the prompts TOF distribution is shown with the randoms TOF distribution (the histogram value for the randoms has been multiplied by 10 to be more visible in the plot). Notice that the randoms, as expected, are uniformly distributed among the nine time bins.

In order to measure SNR at the center of a uniform cylindrical object, we performed two experiments, using two 20-cm-long, uniform water phantoms, with diameters of 20 cm and 37.5 cm. The phantoms were placed roughly in the radial center of the scanner and were injected with $^{18}$F. Several frames were acquired at different activities, in list mode. Each frame corresponded to a different activity and random fraction. Random fraction, defined as the ratio between random coincidences and net true coincidences (prompts minus randoms, or trues+scatter), was computed for all frames.

All frames were acquired for the same time; therefore different activities resulted in different total counts. For a fair comparison of the images, we extracted from each list mode file a subset with the same number of net true coincidences counted in the lowest activity frame, namely $65.5 \times 10^6$ (average 0.21 $10^6$ per plane) for the 20-cm cylinder, and $43.0 \times 10^6$ (average 0.14 $10^6$ per plane) for the 37.5-cm cylinder.

Separate prompt and random sinograms were created from the list mode files. The sinogram dimensions were 336 radial bins, 336 angular bins, 313 planes (divided in 5 segments, 81 were direct planes), and 9 TOF bins. Each TOF bin was 500 ps wide. Since the randoms were distributed uniformly across the TOF bins, the randoms in nine time bins were added, smoothed plane by plane, divided by nine, and placed in the nine bins of the now-smoothed randoms sinogram. The prompts were then corrected by subtracting the smoothed randoms. The resulting net true sinogram was used for reconstruction. For conventional reconstruction, all TOF bins were added.
The TOF and conventional sinograms were then normalized, attenuation-corrected, and scatter-corrected. The standard normalization array and the CT-based attenuation correction were used for both reconstructions. A TOF version [21] of the single-scatter simulation scatter correction was used in this paper for the TOF reconstruction instead of the conventional scatter correction. Only the 81 direct planes were reconstructed into 168 × 168 pixel images (4 mm pixel size), using conventional filtered-back-projection (FBP) with a ramp filter and the new TOF-FBP algorithm described in detail in [3].

The SNR ratio was measured on the reconstructed images. SNR was estimated as the ratio between the average value in a region-of-interest (ROI) and the standard deviation in the same ROI. The ROIs were 12 × 12 pixel squares at the center of the cylinders (equivalent to a 4.8-cm side).

The ratio between SNR_{TOF} and SNR_{conv} was computed for the central 30 planes. The average value over the planes was defined as the TOF SNR gain, and the error associated with it was estimated as the standard deviation over the 30 planes.

IV. RESULTS AND DISCUSSION

In Table I, the TOF SNR gain is reported for the 20-cm cylinder as a function of the random fraction. The corresponding activity and total count rate is also reported. As a reference, the traditional estimate for the TOF gain ($\sqrt{D/\Delta x}$) is reported as a “low random” expected value.

Given the poor time resolution of the scanner, the expected TOF gain is very close to one for a 20-cm-diameter cylinder (1.28 ns resolution corresponds to $\Delta x$ of about 19 cm). Still, the gain is measurable and increases with random fraction in excellent agreement with our model, as can be observed in Fig. 2.

In Table II, the TOF SNR gain is reported for the 37.5-cm cylinder. In this case, the expected gain is much higher, given the larger object. Again, the gain increases with the random fraction and the model is compatible with the experimental data, given the experimental error. The TOF gain versus random fraction is presented in Fig. 3 for the 37.5-cm-diameter cylinder. In this case, the measured gain at low random count rate seems to converge to a lower value than expected. Fitting the data with a function of $R_f$ as in (6), we obtain a “low random” gain of 1.32 instead of the expected 1.4.

Even though some authors argue that (2) is an overestimate of the actual SNR gain we can actually achieve with TOF [16], [22], the results of this work seem to confirm (2) as a valid estimate of low count rate TOF gain. In particular, from the data reported in this paper, there is a perfect agreement with the traditional low count rate TOF gain reported in the literature, in the case of the 20-cm cylinder. And, for the larger object, larger experimental errors due to a lower number of counts, along with higher scatter fraction (which can increase systematic errors due to scatter correction), might be responsible for a 6% mismatch between expected and measured “low random” gain.

In the evaluation and discussion of these results, it should be pointed out that the choice of reconstruction algorithm may affect the measurement of SNR gain. In particular, iterative reconstruction methods are very sensitive to the problem, even though SNR gain is still observed. Because of the different convergence rate of conventional and TOF reconstruction, the choice of iteration number in the two methods is arbitrary, and can lead to different conclusions. In order to reduce the variability due to the reconstruction method, we used an analytical method—the FBP [3]. FBP is also more consistent with the noise propagation.
model used to obtain (6) [8],[9]. Still, different filters are possible for conventional FBP and—more critical—different implementations of TOF FBP have been used in the literature, and it is still possible that the SNR depends on the TOF FBP implementation.

V. Conclusion

The traditional model for SNR gain due to TOF reconstruction was modified to include a more accurate treatment of the contribution from randoms. This model converges to the traditional model at low random fraction values. The SNR gain consistently increases with the random fraction. Experimental data obtained on a LSO PET scanner with (limited) TOF capabilities confirm the model.

Future work will include investigating the phenomenon on TOF PET scanners with improved time resolution, and assess the validity and the limits of the model expressed in (6) with larger objects.

APPENDIX

Brownell and Strother relate the SNR to the variance in an image element \(e\):

\[
\text{SNR} = \text{const} \cdot t_e \cdot \text{VAR}_e^{1/2} \tag{A1}
\]

where \(t_e\) are true coincidence counts in an image element \(e\), and where \(\text{VAR}_e\) is the variance on trues in the image element \(e\), obtained as a weighted variance of the samples from each projection contributing to element \(e\) [10], [11].

If we assume a uniform cylinder of diameter \(D\), and consider the squared image element (of size \(dx^2d\)) at the center of the cylinder, we can estimate that the trues \(t_e\) in the central element are proportional to the size of the image element and the total trues \(T\) in the image:

\[
t_e \approx T \frac{d^2}{\pi (D/2)^2} = T \frac{4d^2}{\pi D^2}. \tag{A2}
\]

This expression is valid for both conventional and TOF reconstructions. The estimate of SNR is therefore reduced to an estimate of the variance \(\text{VAR}_e\), which can be expressed as:

\[
\text{VAR}_e = \sum_{\theta \in e} w_\theta \cdot \text{VAR}_\theta \tag{A3}
\]

where \(\text{VAR}_\theta\) is the variance of the trues of each projection \(\theta\) contributing to element \(e\), and \(w_\theta\) is a weight factor. Trues in projection \(\theta\) are obtained as Prompts-Scatter-Randoms and, if we assume Poisson statistics and that scatter and random estimates have no or negligible variance, the variance on the trues \(\text{VAR}_e\) can be written as

\[
\text{VAR}_e = \sum_{\theta \in e} w_\theta (t_\theta + S_\theta + R_\theta) \tag{A4}
\]

where \(t_\theta, S_\theta\) and \(R_\theta\) are respectively the trues, scatter, and randoms detected in projection \(\theta\) that contribute to the image element \(e\) [11]. If we assume a cylindrical symmetry, the weighted average in (A4) is simply proportional to \((t_\theta + S_\theta + R_\theta)\), as in

\[
\text{VAR}_e = \sum_{\theta \in e} w_\theta (t_\theta + S_\theta + R_\theta) = \text{const} \cdot (t_\theta + S_\theta + R_\theta) \tag{A5}
\]

where \(t_\theta, S_\theta\), and \(R_\theta\) are different in the case of conventional and TOF reconstructions. These are evaluated in the following sections. We use the following conventions: \(D\) is the diameter of the image object, \(d\) is the image element size, \(D_{\text{FOV}}\) is the diameter of the scanner FOV as defined by the coincidence window \((D_{\text{FOV}} = c t_w/2)\), and \(\Delta x = (\Delta t)/(2)c\), where \(\Delta t\) is the time resolution.

A. Trues Variance in Conventional Reconstruction

\[
\text{VAR}_e(\text{conv}) = \text{const} \cdot (t_\theta + S_\theta + R_\theta)_\text{conv}. \tag{A6}
\]

The different terms to be added can be estimated as follows:

I) \(t_\theta = T(4d^2)/(\pi D^2) \cdot (D)/(d)\), the average trues in an image element times the number of object image elements along the projection \(\theta\);

II) \(S_\theta = S(4d^2)/(\pi D^2) \cdot (D)/(d)\), the average scatter in an image element times the number of object image elements along the projection \(\theta\), assuming that the scatter is uniform in a cylinder of diameter \(D\), and there is no scatter outside the object;

III) \(R_\theta = R(4d^2)/(\pi D_{\text{FOV}}^2) \cdot (D_{\text{FOV}})/(d)\), the average randoms in an image element times the number of image elements along the projection \(\theta\), assuming that the randoms are uniform in the whole FOV.

B. Trues Variance in TOF Reconstruction

\[
\text{VAR}_e(\text{TOF}) = \text{const} \cdot (t_\theta + S_\theta + R_\theta)_{\text{TOF}}. \tag{A7}
\]

The different terms to be added can be estimated as follows:

I) \(t_\theta = T(4d^2)/(\pi D^2) \cdot (\Delta x)/(d)\), the average trues in an image element times the number of image elements that contribute to that projection, determined by the TOF resolution or spatial uncertainty \(\Delta x\);

II) \(S_\theta = S(4d^2)/(\pi D^2) \cdot (\Delta x)/(d)\), the average scatter in an image element times the number of image elements that contribute to that projection, determined by the TOF resolution or spatial uncertainty \(\Delta x\), assuming that the scatter is uniform in the object of diameter \(D\) and there is no scatter outside the object;

III) \(R_\theta = R(4d^2)/(\pi D_{\text{FOV}}^2) \cdot (\Delta x)/(d)\), the average randoms in an image element times the number of image elements that contribute to that projection, determined by the TOF resolution or spatial uncertainty \(\Delta x\). We assume that the randoms are uniform in the whole FOV.
Using $t_e$ from equation (A2), the estimate for the variance $\text{VAR}_e$ for the conventional reconstruction (terms I, II, III of (A6)), the estimate for the $\text{VAR}_e$ for the TOF reconstruction (terms I, II, III of (A7)), and substituting in (A1), we obtain

$$\text{SNR}^2(\text{conv}) = \frac{4}{\pi} \left( \frac{d}{D} \right)^3 \frac{T^2}{T + S + \frac{P}{D_{\text{CON}}} R}$$  \hspace{1cm} (A8)$$

and

$$\text{SNR}^2(\text{TOF}) = \frac{4}{\pi} \left( \frac{d}{D} \right)^3 \frac{D}{\Delta x} \frac{T^2}{T + S + \left( \frac{P}{D_{\text{CON}}} \right)^2 R}$$  \hspace{1cm} (A9)$$

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REFERENCES